Measurements.

- Measurements in comp. basis
- Complete measurements

Today: Projective measurement

Q system with $N=3$

Q state $|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

Measurement:

Are we (a) in possibility 1
or (b) in possibility 2/3?

Step 1 Write subspace of Q states corresponding to each answer.

$$V_a = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

$$V_a = \mathrm{span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \right\} = \left\{ \begin{pmatrix}\alpha\\0\\0\end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

$$V_b = \mathrm{span}\left\{ \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \cancel{\begin{pmatrix}0\\1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}} \right\}$$

$$= \left\{ \begin{pmatrix}0\\\beta\\\gamma\end{pmatrix} : \beta, \gamma \in \mathbb{C} \right\}$$

Remember: We measure

$$|\psi\rangle = \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$|\psi\rangle = \underbrace{|\psi_a\rangle}_{\in V_a} + \underbrace{|\psi_b\rangle}_{\in V_b}, \quad |\psi_a\rangle = \begin{pmatrix}1/\sqrt{3}\\0\\0\end{pmatrix}, \quad |\psi_b\rangle = \begin{pmatrix}0\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

(In general: Need that

$\sum_i V_i = \mathcal{H} \hat{=}$ whole space,

V_i, V_j orthogonal)

Step 2: Get the outcome probs.

$$|\psi_a\rangle = \begin{pmatrix} 1/\sqrt{3} \\ 0 \\ 0 \end{pmatrix} \qquad |\psi_b\rangle = \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$P[\text{outcome } (a)] = \| |\psi_a\rangle \|^2$$
$$= \left| 1/\sqrt{3} \right|^2 = \frac{1}{3}$$

$$P[\text{outcome } (b)] = \| |\psi_b\rangle \|^2$$
$$= \left| \frac{1}{\sqrt{3}} \right|^2 + \left| \frac{1}{\sqrt{3}} \right|^2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Step 3 Post-meas-state:

After outcome (a): $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\frac{|\psi_a\rangle}{\| |\psi_a\rangle \|} = \frac{\begin{pmatrix} 1/\sqrt{3} \\ 0 \\ 0 \end{pmatrix}}{1/\sqrt{3}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

After outcome (b):

$$\frac{|\psi_b\rangle}{\| |\psi_b\rangle \|} = \frac{\begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}}{2/\sqrt{3}} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

<u>Def:</u> Proj. Measurement
(in a Q system $\mathcal{H} = \mathbb{C}^N$)

Consists of: finite
family V_i $(i \in I)$ of
subspaces of $\mathcal{H}$
<u>s.t.</u> $\sum V_i = \mathcal{H}$, V_i, V_j orthog $(i \neq j)$

(I = set of outcomes)

$\Pr[\text{outcome } i] = \| |\psi_i\rangle \|^2$
where $|\psi\rangle = \sum_i |\psi_i\rangle$, $|\psi_i\rangle \in V_i$
post meas-state for i:
$|\psi_i\rangle / \| |\psi_i\rangle \|$

Complete meas. are special
cases of proj. meas:
$|\psi_1\rangle, \dots, |\psi_n\rangle \rightsquigarrow V_1, \dots, V_n$
$V_i = \text{span}\{|\psi_i\rangle\}$

Proj. meas. in terms of projectors:

Projector P

$\Leftrightarrow$ $P^2 = P$

$P^+ = P$

Projectors on $\mathcal{H}$ stand in 1-1 correspondence with subspaces:

- $V \rightsquigarrow$ proj. onto V
- $P \rightsquigarrow \operatorname{im} P$

<u>Def</u>: <u>Proj. meas.</u>

Consists of family P_i $(i \in I)$ of proj. s.t. $\sum P_i = 1$, $P_i P_j = 0$ $(i \neq j)$

$$\Pr[\text{outcome } i] = \| P_i |\psi\rangle \|^2$$

post-meas-state: $\dfrac{P_i |\psi\rangle}{\| P_i |\psi\rangle \|}$

Back to the example

$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$P_a = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad \left(\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \to \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\ 0\ 0)$$

$$P_b = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \left(\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \to \begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (0\ 1\ 0) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (0\ 0\ 1)$$

Given $|\phi\rangle$ (normalized)
proj onto span$\{|\phi\rangle\}$
$= |\phi\rangle \cdot |\phi\rangle^\dagger$
$= |\phi\rangle\langle\phi|$
$= |\phi\rangle\!\langle\phi|$

Given $|\phi_1\rangle, \ldots, |\phi_n\rangle$
(norm, orthog):
proj onto span{those}
$= \sum_i |\phi_i\rangle\!\langle\phi_i|$

Pr[outcome (a)]
$= \| P_a |\psi\rangle \|^2$
$= \left\| \begin{pmatrix} 1/\sqrt{3} \\ 0 \\ 0 \end{pmatrix} \right\|^2 = \frac{1}{3}$

Pr[outcome (b)] =
$\| P_b |\psi\rangle \|^2 = \left\| \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right\|^2 = \frac{2}{3}$

pms : $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

Complete meas as proj. meas

Compl. meas: $|\varphi_1\rangle, \ldots, |\varphi_n\rangle$
(orthog, norm)

Corresponding proj. meas:

$P_i :=$ proj onto span $\{|\phi_i\rangle\}$
$= |\phi_i\rangle\langle\phi_i|$

According to compl meas	According to proj. m.
$\Pr[\text{outcome } i]$ $= \lvert\alpha_i\rvert^2$ where: $\lvert\psi\rangle = \alpha_1\lvert\phi_1\rangle + \ldots + \alpha_n\lvert\phi_n\rangle$	$\Pr[\text{outcome } i]$ $= \lVert\, \lvert\phi_i\rangle\langle\phi_i\lvert\psi\rangle \rVert^2$ $= \lvert\langle\phi_i\lvert\psi\rangle\rvert^2$ $= \lvert\alpha_i\rvert^2$
p-m-s for i: $\lvert\phi_i\rangle$	p-m-s: $\dfrac{\lvert\phi_i\rangle\langle\phi_i\lvert\psi\rangle}{\lvert\langle\phi_i\lvert\psi\rangle\rvert}$ $= \dfrac{\langle\phi_i\lvert\psi\rangle}{\lvert\langle\phi_i\lvert\psi\rangle\rvert} \cdot \lvert\phi_i\rangle$

$\langle\phi|\cdot|\xi\rangle$
$= \langle\phi, \xi\rangle$